Supersymmetry in Spaces of Constant Curvature

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1 Abstract

Supersymmetry is considered in spaces of constant curvature (spherical, de Sitter and Anti-de Sitter spaces) of two, three and four dimensions.

2 Introduction

The expectation that supersymmetry will soon be shown to be a symmetry of nature motivates us to analyze its properties in spaces of constant curvature. We view these D dimensional spaces as being surfaces, in a flat embedding space of dimension $D+1$, that satisfy the (constraint) equation

$$g_{AB}x^A x^B = \text{const.} \tag{1}$$

For spherical space times S_D, $g_{AB} = \text{diag}(+,+,+,\ldots+)$, for de Sitter space dS_D, $g_{AB} = \text{diag}(+,+,+,\ldots+,-)$ and for anti de Sitter space, AdS_D, $g_{AB} = \text{diag}(+++\ldots+,-,-)$. The symmetry (or isometry) transformations on this surface are generated by operators J^{AB} which satisfy the algebra

$$\left[J^{AB}, J^{CD}\right] = g^{AC}J^{BD} - g^{BC}J^{AD} + g^{BD}J^{AC} - g^{AD}J^{BC}. \tag{2}$$

(We use anti-Hermitian generators.) Depending on the metric, this is the algebra of the groups $S0(D+1)$, $S0(D,1)$ and $S0(D-2,2)$ for S_D, dS_D and AdS_D respectively. We do not necessarily include Bosonic translation generators P_μ, in contrast to ref [1].

The isometry group of flat space-times is a semi-direct product of an SO group, such as the above, with the Abelian group of translations; for example in $3+1$ dimensions the

Poincarë group is $SO(3,1) \times T_4$.

The supersymmetry generators Q in flat space-time occur as "square roots" of the translation generators P ($\{Q, Q\} \sim P$). The translation generators P^A commute with the supersymmetry generators Q and the SO generators J^{AB}.

In the case of the spaces of constant curvature translations are not isometries. The supersymmetry generators Q must then occur as "square roots" of the J^{AB} ($\{Q, Q\} \sim J^{AB}$); the J^{AB} do not commute with Q. Closure of the algebra requires in many cases extra Bosonic generators. These do not necessarily commute with everything and consequently are not "central charges" but rather "internal symmetry generators". This is in contrast to the central charges considered in [2].

In section 3 we consider a number of different possible supersymmetry algebras which extend the algebra of (2) for these spaces of constant curvature for $D = 4, 3$ and 2. In each case we have identified extensions with the least possible number of Fermionic generators. (We consider cases in which the simplest Fermionic generator is a Dirac spinor, unlike ref. [1].)

In Sections 4, 5 and 6 we focus our attention on the simplest supersymmetry algebras in the spaces S_2 and AdS_2. In Section 4 we examine the representations of the algebras. For S_2 we classify the states which carry an irreducible representation of the algebra and we show that there is an upper bound on the angular momentum for these states. For AdS_2 we generalise the previous treatment of the $N = 1$ supersymmetry algebra [8, 9, 16] to the $N = 2$ case. The $N = 2$ algebra in AdS_2 resembles closely the simplest supersymmetry algebra in S_2. In section 5 we provide examples of supersymmetric models on S_2 and AdS_2 containing interacting scalar and spinor fields. In section 6 we provide a realization in appropriate superspaces of the minimal supersymmetry algebra in S_2 and AdS_2 using the Bosonic coordinates of the 3-dimensional embedding spaces. In the AdS_2 case we define scalar superfields and we write down a number of distinct supersymmetric actions in terms of them. In terms of the component fields these actions correspond to realistic models on AdS_2.

Notation and Dirac matrix identities are given in an appendix.

3 Supersymmetry Algebra

3.1 S_4

As noted after eq. (A.36), in $5+0$ dimensions the simplest spinor is necessarily Dirac[3]; we consequently employ a Dirac spinorial generator Q_i. It is necessary to include a scalar internal symmetry generator Z in addition to J^{AB} in order to define a closed supersymmetric extension of the algebra of eq. (2) for S_4. The two superalgebras whose non-vanishing (anti-)commutators are

$$\left[J^{AB}, Q_i\right] = -\Sigma_{ij}^{AB} Q_j \tag{3a}$$

$$[Z, Q_i] = \mp Q_i \tag{3b}$$

$$\left\{Q_i, Q_j^\dagger\right\} = Z\delta_{ij} \pm \Sigma_{ij}^{AB} J^{AB} \tag{3c}$$

satisfy the Jacobi identities. (Eq. (A.34) is useful in proving this.)

Another pair of superalgebras associated with S_4 is given by

$$\left[J^{AB}, Q_i\right] = -\Sigma_{ij}^{AB} Q_j \qquad \left[J^{AB}, Z^C\right] = \delta^{AC} Z^B - \delta^{BC} Z^A \tag{4a, b}$$

$$\left[Z^A, Q_i\right] = -\tfrac{1}{2}\gamma_{ij}^A Q_j \qquad [Z, Q_i] = -Q_i \tag{4c, d}$$

$$\left[Z^A, Z^B\right] = -J^{AB} \tag{4e}$$

$$\left\{Q_i, Q_j^\dagger\right\} = \pm\left(\tfrac{3}{2} Z\delta_{ij} - \gamma_{ij}^A Z^A + \Sigma_{ij}^{AB} J^{AB}\right). \tag{4f}$$

It too is consistent with the Jacobi identities. The superalgebras of equations (4) differ from those of equations (3) through the inclusion of an $SO(5)$ vector bosonic generator Z^A which does not commute with J^{AB} and a distinct anti-commutator $\left\{Q, Q^\dagger\right\}$. Z^A plays the role of a translation operator as in ref. [1]. In (3) and (4), the J_{AB} are anti-Hermitian while Z_A and Z are Hermitian.

3.2 S_3

In $4+0$ dimensions also the simplest spinors are Dirac [3]. Consequently one can obtain a superalgebra associated with S_3 by "dimensional reduction" of the superalgebras of eq. (3) and eq. (4). One identifies the generators $J^{\alpha 5}$ with an $S0(4)$ vector generator Y^α. The superalgebra of (3) is then rewritten as

$$\left[J^{\alpha\beta}, Q_i\right] = -\Sigma_{ij}^{\alpha\beta} Q_j, \qquad [Y^\alpha, Q_i] = \tfrac{1}{2}\left(\gamma^\alpha\gamma_5\right)_{ij} Q_j \tag{5a, b}$$

$$\left[J^{\alpha\beta}, Y^\gamma\right] = \left(\delta^{\alpha\gamma} Y^\beta - \delta^{\beta\gamma} Y^\alpha\right), \qquad \left[Y^\alpha, Y^\beta\right] = J^{\alpha\beta} \tag{5c, d}$$

$$[Z, Q_i] = \mp Q_i, \quad \left\{Q_i, Q_j^\dagger\right\} = Z\delta_{ij} \pm \left(\Sigma_{ij}^{\alpha\beta} J^{\alpha\beta} - \tfrac{1}{2}\left(\gamma^\alpha\gamma_5\right)_{ij} Y^\alpha\right) \tag{5e, f}$$

in addition to eq. (2). Again, Y is a translation operator, as in [1].

When "dimensionally reducing" the superalgebra of eq. (4), we again identify $J^{\alpha 5}$ with Y^{α}; in addition Z^5 becomes the $S0(4)$ scalar Y. It is straightforward to effect the "dimensional reduction" of the superalgebra of eq. (4); the approach employed in obtaining the superalgebra of eq. (5) from that of eq. (3) is followed.

Of more interest, there are two superalgebras associated with S_3 which cannot be obtained by "dimensional reduction". These superalgebras are the analogues of equations (3). They require two internal symmetry generators Z and Z_5 to satisfy the Jacobi Identities. The superalgebras are given by

$$\left\{Q_i, Q_j^{\dagger}\right\} = \mp\Sigma_{ij}^{\alpha\beta} J^{\alpha\beta} + Z\delta_{ij} + Z_5 \left(\gamma_5\right)_{ij} \tag{6a}$$

$$[Z, Q_i] = \pm\tfrac{1}{2}Q_i \quad [Z_5, Q_i] = \pm\tfrac{1}{2}(\gamma_5)_{ij}Q_j \tag{6b, c}$$

$$\left[J^{\alpha\beta}, Q_i\right] = -\Sigma_{ij}^{\alpha\beta} Q_j \tag{6d}$$

in addition to eq. (2). This superalgebra can be further decomposed into two decoupled subalgebras, using the chiral decomposition of the generators Q into $\frac{1}{2}\left(1 \pm \gamma_5\right) Q$.

3.3 S_2

Superalgebras with one Fermionic generator associated with a two dimensional spherical surface embedded in $3+0$ dimensions cannot be obtained by dimensional reduction of the superalgebras associated with S_3, as in $3+0$ dimensions irreducible spinors are two component Dirac spinors while in $4+0$ dimensions they are four component Dirac spinors.

Superalgebras, with one Fermionic generator, similar in form to those of (3) and (6) can however be written down on S_3. If Q_i is a two-component Dirac spinorial generator, then we find two superalgebras consistent with the Jacobi identities, namely

$$\left\{Q_i, Q_j^\dagger\right\} = Z\delta_{ij} \mp 2\tau_{ij}^a J^a \tag{7a}$$

$$[J^a, Q_i] = -\tfrac{1}{2}\tau_{ij}^a Q_j \tag{7b}$$

$$[Z, Q_i] = \mp Q_i \tag{7c}$$

$$\left[J^a, J^b\right] = i\epsilon^{abc} J^c. \tag{7d}$$

In addition there is in two dimensions a third superalgebra associated with S_2. It can be shown that (with $\tilde{Q} = Q^T\tau_2$)

$$\left\{Q_i, \tilde{Q}_j\right\} = \tau_{ij}^a J^a \qquad \left\{Q_i, Q_j^\dagger\right\} = \tau_{ij}^a Z^a \tag{8a, b}$$

$$[J^a, Q_i\,,] = -\tfrac{1}{2}\tau_{ij}^a Q_j \qquad \left[Z^a, \tilde{Q}_i\right] = \tfrac{1}{2}Q_j^\dagger \tau_{ji}^a \tag{8c, d}$$

$$\left[J^a, J^b,\right] = i\epsilon^{abc} J^c \qquad \left[Z^a, Z^b\right] = -i\epsilon^{abc} J^c \tag{8e, f}$$

$$\left[J^a, Z^b\right] = i\epsilon^{abc} Z^c \tag{8g}$$

is consistent with Jacobi identities. This algebra does not appear to have an analogue on S_3 or S_4. If we define the symplectic Majorana spinors

$$Q_1 = \frac{Q + Q_c}{2} = -(Q_2)_c$$

$$Q_2 = \frac{Q - Q_c}{2} = (Q_1)_c$$

then it follows from (8a,b) that

$$\left\{Q_1, \tilde{Q}_2\right\} = \left\{Q_2, \tilde{Q}_1\right\} = \tfrac{1}{2}\tau^a J^a$$

$$\left\{Q_1, \tilde{Q}_1\right\} = -\left\{Q_2, \tilde{Q}_2\right\} = -\tfrac{1}{2}\tau^a Z^a$$

Z^a is akin to the translation operator appearing in [1]. A similar decomposition of (7) (with the upper sign) leads to

$$\left\{Q_i, \tilde{Q}_j\right\} = -\tau^a_{ij} J^a + \frac{1}{2} Z \epsilon_{ij}$$

3.4 dS_4

By exploiting some of the properties of spinors in 4+1 dimensions (as given in the appendix) we are able to formulate a supersymmetric algebra associated with dS_4. We introduce two 4-component Dirac spinorial generators Q^r_i $(r = 1, 2)$ related by the symplectic Majorana condition (A.15),

$$Q^r_i = \epsilon^{rs} Q^y_{s\,i}. \tag{9}$$

The generator $\tilde{Q}^r_i$ is defined to be

$$\tilde{Q}^r_i = \left(Q^{Tr} C\right)_i \tag{10}$$

with $C = C^\dagger = C^{-1} = -C^T = -C^*$ given by eq. (A.38). It is evident that $\left(\tilde{Q}^r Q^s\right)$ is a Lorentz scalar in $4+1$ dimensions.

A consistent superalgebra employing these spinors that involves a translation operator Z_A is given by

$$\left\{Q^r_i, \tilde{Q}^s_j\right\} = i\left[\delta^{rs} \Sigma^{AB}_{ij} J_{AB} + \epsilon^{rs}\left(\gamma^A_{ij} Z_A + \delta_{ij} Z\right)\right] \tag{11a}$$

$$\left[J^{AB}, Q_i^r\right] = -\left(\Sigma^{AB} Q^r\right)_i \tag{11b}$$

$$\left[Z^A, Q_i^r\right] = -\tfrac{1}{2}\epsilon^{rs}\left(\gamma^A Q^s\right)_i \tag{11c}$$

$$\left[Z, Q_i^r\right] = \tfrac{3}{2}\epsilon^{rs} Q_i^s \tag{11d}$$

$$\left[Z^A, Z^B\right] = J^{AB} \tag{11e}$$

$$\left[J^{AB}, Z^C\right] = g^{AC} Z^B - g^{BC} Z^A. \tag{11f}$$

Showing that the Jacobi identities are satisfied involves using the properties of Dirac matrices (A.28 - A.34). In addition, to analyze the Jacobi identity associated with $\left(Q_i^r, Q_j^s, Q_k^t\right)$, (A.41) is useful. The algebra of eq. (11) is closed under Hermitian conjugation with the symplectic Majorana condition (9), provided $J_{AB}^\dagger = -J_{AB}$, $Z_A^\dagger = -Z_A$ and $Z^\dagger = -Z$. Of course, one could always use a single 4-component Dirac spinor in place of the symplectic Majorana spinors (9). In that case, an analogue of either (3) and (4) could be introduced in dS_4; similarly (11) has an analogue on S_4. A mapping between these two algebras in the latter case between the algebras of (11) and (4) is provided by $\dfrac{Q^1 + iQ^2}{\sqrt{2}} \to Q$, $iZ^A \to Z^A$, $Z \to \frac{3i}{2} Z$.

3.5 AdS_4

In the case of AdS_4, one can use spinorial generators which are Majorana rather than Dirac, as the Majorana condition can be consistently applied in $3+2$ dimensions.

If now $\tilde{Q} = Q^T C$ with C defined by eq. (A.40), then we can have

$$\left\{Q_i, \tilde{Q}_j\right\} = 2\Sigma_{ij}^{AB} J_{AB} \tag{12a}$$

$$\left[J^{AB}, Q_i\right] = -\left(\Sigma^{AB} Q\right)_i \tag{12b}$$

as a consistent supersymmetric generalization of the algebra associated with AdS_4. Again, (A.41) is crucial for establishing that the Jacobi identities are satisfied.

We note that (12a) is consistent in the sense that $\left(\Sigma^{AB}C\right)^T = \Sigma^{AB}C$; vectorial and scalar internal symmetry generators cannot consistently be introduced into (12a) as $\left(\gamma^A C\right)^T = -\gamma^A C$, $C^T = -C$.

3.6 dS_3

Using the notation employed in the appendix we note that the usual supersymmetry algebra for $3+1$ dimensional Minkowski space, is given by

$$\left\{Q_i, \tilde{Q}_j\right\} = 0 \qquad \left\{Q_i, Q_j^\dagger\right\} = 2\sigma_{ij}^\lambda P_\lambda$$

$$\left[P_\mu, \tilde{Q}_1\right] = 0 \qquad \left[J_{\mu\nu}, Q_i\right] = -\left(\sigma_{\mu\nu} Q\right)_i \qquad \left[J_{\mu\nu}, Q_i^\dagger\right] = \left(Q^\dagger \overline{\sigma}_{\mu\nu}\right)_i \tag{13a}$$

$$\left[P_\mu, P_\nu\right] = 0 \quad \left[J_{\mu\nu}, P_\lambda\right] = g_{\mu\lambda} P_\mu - g_{\nu\lambda} P_\mu \quad \left[J_{\mu\nu}, J_{\lambda\sigma}\right] = g_{\mu\lambda} J_{\nu\sigma} - g_{\mu\sigma} J_{\nu\lambda} + g_{\nu\sigma} J_{\mu\lambda} - g_{\nu\lambda} J_{\mu\sigma}.$$

Since no anticommutator of the form $\left\{Q_i, \tilde{Q}_j\right\}$, $\left\{Q_i, Q_j^\dagger\right\}$ or $\left\{Q_i^\dagger, Q_j^\dagger\right\}$ can consistenty be related to $J_{\mu\nu}$, this algebra cannot be viewed as a superalgebra in dS_3. However, as in [1] if we relax the condition $[P_\mu, P_\nu] = 0$ and identify P_μ with an internal symmetry generator Z_μ, then we can consistently relate $\left\{Q_i, \tilde{Q}_j\right\}$ and $\left\{\tilde{Q}_i^\dagger, Q_j^\dagger\right\}$ to $J_{\mu\nu}$ so that the resulting superalgebra can be associated with dS_3. It is

$$\left\{Q_i, \tilde{Q}_j\right\} = -2\sigma_{ij}^{\mu\nu} J_{\mu\nu}, \quad \left\{\tilde{Q}_i^\dagger, Q_j^\dagger\right\} = -2\overline{\sigma}_{ij}^{\mu\nu} J_{\mu\nu} \quad \left\{Q_i, Q_j^\dagger\right\} = 2\sigma_{ij}^\lambda Z_\lambda$$

$$\left[J_{\mu\nu,}, Q_i\right] = -\left(\sigma_{\mu\nu} Q\right)_i \qquad \left[J_{\mu\nu,} Q_i^\dagger\right] = \left(Q^\dagger \overline{\sigma}_{\mu\nu}\right)_i$$

$$\left[Z_\mu, \tilde{Q}_i\right] = \tfrac{1}{2}\left(Q^\dagger \overline{\sigma}_\mu\right)_i \qquad \left[Z_\mu, Q_i^\dagger\right] = \tfrac{1}{2}\left(\tilde{Q}\sigma_\mu\right)_i \tag{13b}$$

$$[Z_\mu, Z_\nu] = -J_{\mu\nu} \quad [J_{\mu\nu}, Z_\lambda] = g_{\mu\lambda}Z_\nu - g_{\nu\lambda}Z_\mu \quad [J_{\mu\nu}, J_{\lambda\sigma}] = g_{\mu\lambda}J_{\nu\sigma} - g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\sigma}J_{\mu\lambda} - g_{\nu\lambda}J_{\mu\sigma}$$

It can easily be shown that all Jacobi identities associated with the triples (Q,Q,Q), $(Q,Q,Q^\dagger)$, (P,Q,Q), $(P,Q,Q^\dagger)$, (P,P,Q), (J,P,Q), (P,P,J) and (P,J,J) are satisfied by the algebra of (13b). This can be viewed as a supersymmetric extension of the algebra considered in [4].

3.7 AdS_3

In $2+2$ dimensions, a spinorial generator Q can be simultaneously Majorana and Weyl. For the simplest supersymmetric extension of the isometry algebra of AdS_3, we consider a spinorial generator in $2+2$ dimensions which is Majorana-Weyl. Using the notation of [5] with

$$(\sigma^{\mu\nu})_{k\ell} = -\tfrac{1}{4}\left[\sigma^\mu{}_{k\dot{m}}\overline{\sigma}^{\nu\dot{m}}{}_\ell - \sigma^\nu{}_{k\dot{m}}\overline{\sigma}^{\mu\dot{m}}{}_\ell\right] \tag{14}$$

we find that

$$\left\{Q_k, Q^\ell\right\} = (\sigma^{\mu\nu})_k{}^\ell J_{\mu\nu} \tag{15a}$$

$$[J^{\mu\nu}, Q_k] = -(\sigma^{\mu\nu})_k{}^\ell Q_\ell \tag{15b}$$

is a superalgebra which is consistent with the Jacobi identities.

3.8 AdS_2/dS_2

There is a degeneracy between two dimensional Anti-de Sitter space and two dimensional de Sitter space. We begin by relating supersymmetry in AdS_2 space to superconformed symmetry in $0+1$ dimensions [6]. The $N=2$ superconformal algebra in $0+1$ dimensions is given by [7-9]

$$[\pi,\delta]=\pi \qquad [\kappa,\delta]=-\kappa \qquad [\pi,\kappa]=2\delta \tag{16a}$$

$$[\delta,q_i]=-\tfrac{1}{2}q_i \qquad [\delta,s_i]=\tfrac{1}{2}s_i \tag{16b}$$

$$[\pi,s_i]=iq_i \qquad [\kappa,q_i]=is_i \tag{16c}$$

$$\{q_i,q_j\}=\pm i\delta_{ij}\pi \qquad \{s_i,s_j\}=\mp i\delta_{ij}\kappa \tag{16d}$$

$$\{q_i,s_j\}=\mp\delta_{ij}\delta-\tfrac{i}{2}\epsilon_{ij}\alpha \tag{16e}$$

$$[s_i,\alpha]=\mp i\epsilon_{ij}s_j \qquad [q_i,\alpha]=\mp i\epsilon_{ij}q_j. \tag{16f}$$

(This can be derived by projecting the $N=2$ superconformal algebra in $1+1$ dimensions along the light cone [9].) By setting $\alpha=q_2=s_2=0$ in (16), a consistent $N=1$ version of this superconformal algebra can be obtained. We note that $i\delta$, π, κ, α, $\sqrt{i}q_i$ and $\sqrt{-i}s_i$ are Hermitian.

The generators π, δ and κ ("Hamiltonian", "dilitation" and "special conformal" generators respectively) have a relationship with those of AdS_2 space given in [6]. We follow this prescription, defining the symmetry generators $J_{ab}(=-J_{ba})$ in AdS_2 space by

$$J_{12}=\tfrac{1}{2}(\kappa-\pi) \quad J_{13}=\tfrac{1}{2}(\kappa+\pi) \quad J_{32}=\delta. \tag{17}$$

In addition, we define the two component spinor

$$Q=\begin{pmatrix} q+is \\ q-is \end{pmatrix}. \tag{18}$$

The $N = 1$ limit of supersymmetry algebra of (16) implies that

$$[J_{ab}, J_{cd}] = g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}$$

$$\left\{Q, \tilde{Q}\right\} = 2\Sigma^{ab}J_{ab} \tag{19}$$

$$[J_{ab}, Q] = -\Sigma_{ab}Q.$$

If now we look at the full $N = 2$ superalgebra of (16) with

$$Q_i = \begin{pmatrix} q_i + is_i \\ q_i - is_i \end{pmatrix} \tag{20}$$

then (16) implies that

$$\left\{Q_i, \tilde{Q}_j\right\} = -i\epsilon_{ij}\alpha \pm 2\delta_{ij}\Sigma^{ab}J_{ab}$$

$$[J_{ab}, Q_i] = -\Sigma_{ab}Q_i \tag{21}$$

$$[\alpha, Q_i] = \pm i\epsilon_{ij}Q_j.$$

From (18) and (20) it is apparent that the spinor generators in the algebras (19) and (21) each possess two degrees of freedom, and so are not true Dirac spinors. In the algebras (19) and (21) Q and Q_i can be interpreted as being Majorana spinors. The two Majorana spinors in (21) can be combined to form a Dirac spinor

$$Q = Q_1 + iQ_2\ ; \tag{22}$$

in this case (21) becomes

$$\left\{Q, \tilde{Q}\right\} = 0$$

$$\left\{Q, \overline{Q}\right\} = \mp 4\Sigma^{ab}J_{ab} + 2\alpha \tag{23}$$

$$[J_{ab}, Q] = -\Sigma_{ab} Q$$

$$[\alpha, Q] = \pm Q.$$

Using a Dirac spinor generator Q, we can formulate another consistent superalgebra that is an extension of the AdS_2 algebra.

$$\left\{Q, \tilde{Q}\right\} = \Sigma^{ab} J_{ab} \qquad \left\{Q, \overline{Q}\right\} = \Sigma^{ab} Z_{ab}$$

$$\left[J^{ab}, Q\right] = -\Sigma^{ab} Q \qquad \left[Z^{ab}, \tilde{Q}\right] = \overline{Q}\Sigma^{ab} \tag{24}$$

$$\left[J^{ab}, J^{cd}\right] = g^{ac} J^{bd} - g^{bc} J^{ad} + g^{bd} J^{ac} - g^{ad} J^{bc}$$

$$\left[Z^{ab}, Z^{cd}\right] = g^{ac} J^{bd} - g^{bc} J^{ad} + g^{bd} J^{ac} - g^{ad} J^{bc}$$

$$\left[J^{ab}, Z^{cd}\right] = g^{ac} Z^{bd} - g^{bc} Z^{ad} + g^{bd} Z^{ac} - g^{ad} Z^{bc}.$$

This superalgebra is analogous to the S_2 superalgebra of eq. (8). It can be shown to satisfy the Jacobi identities. On AdS_2, Q can be decomposed into two Majorana spinors

$$Q_{1,2} = \frac{Q \pm Q_c}{2}. \tag{25}$$

The algebra of (24), in terms of $Q_{1,2}$, becomes

$$\left\{Q_1, \tilde{Q}_2\right\} = 0$$

$$\left\{Q_{1,2}, \tilde{Q}_{1,2}\right\} = -\tfrac{1}{2}\Sigma^{ab}\left(Z_{ab} \mp J_{ab}\right) \tag{26}$$

$$\left[J^{ab}, Q_{1,2}\right] = -\Sigma^{ab} Q_{1,2}$$

$$\left[Z^{ab}, Q_{1,2}\right] = \pm\Sigma^{ab} Q_{1,2}$$

$$\left[K_{\pm}^{ab}, K_{\pm}^{cd}\right] = g^{ac}K_{\pm}^{bd} - g^{bc}K_{\pm}^{ad} + g^{bd}K_{\pm}^{ac} - g^{ad}K_{\pm}^{bc}$$

$$\left[K_{\pm}^{ab}, K_{\mp}^{cd}\right] = 0$$

where $K_{\pm}^{ab} = \frac{1}{2}\left(J^{ab} \pm Z^{ab}\right)$. It is evident from (26) that Q_1 and Q_2 both belong to a subalgebra with the structure of eq. (19).

4 Representations in Two Dimensions

4.1 S_2

The analysis of the representations of the superalgebra of eq. (7) which is associated with S_2 closely follows the discussion of the $sp\ell(2,1)$ superalgebra in [10]. (The $osp(2,1)$ superalgebra considered in [10] is not self-adjoint.) We first note that there are two Casimirs associated with (7) (with the upper sign)

$$\begin{aligned} \not{C}_2 &= \vec{J}^2 - \tfrac{1}{4}Z^2 - \tfrac{1}{2}Z + \tfrac{1}{2}Q^\dagger Q \\ &= \vec{J}^2 - \tfrac{1}{4}Z^2 - \tfrac{1}{2}\left[Q_i, Q_i^\dagger\right] \end{aligned} \tag{27}$$

and

$$\not{C}_3 = \tfrac{1}{2}(Z+1)\left(\not{C}_2 + \tfrac{1}{2}Q^\dagger Q\right) + \tfrac{1}{4}Q^\dagger \tau \cdot JQ. \tag{28}$$

The subalgebra of (7d) has the usual Casimir $\vec{J}^2$.

If $|I >= |j, m, \zeta >$ with

$$\vec{J}^2|I >= j(j+1)|I > \tag{29a}$$

$$J_3|I >= m|I > \tag{29b}$$

$$Z|I>=\zeta|I> \tag{29c}$$

subject to the requirement

$$Q_i|I>=0 \quad (i=1,2), \tag{30}$$

we then define

$$Q_i^\dagger|I>=|i> \quad (i=1,2) \tag{31a}$$

$$Q_1^\dagger Q_2^\dagger|I>=|F>. \tag{31b}$$

From (7) it follows that

$$J_3|1>=\left(m-\tfrac{1}{2}\right)|1> \tag{32a}$$

$$J_3|2>=\left(m+\tfrac{1}{2}\right)|2> \tag{32b}$$

$$J_3|F>=m|F>; \tag{32c}$$

furthermore

$$Z|i>=(\zeta\pm 1)|i> \tag{33a}$$

$$Z|F>=(\zeta\pm 2)|F>. \tag{33b}$$

(The two signs in (33) correspond to the two algebras in (7).)

It is possible to show that

$$\left[J^2, Q_1^\dagger Q_2^\dagger\right]=0 \tag{34}$$

and hence

$$J^2|F>=j(j-1)|F>; \tag{35}$$

however, $|i>$ is a linear combination of states which are eigenfunctions of J^2 corresponding to eigenvalues $(j+\frac{1}{2})(j+\frac{3}{2})$ and $(j-\frac{1}{2})(j+\frac{1}{2})$.

(The operators appearing in (7) are related to those appearing in the discussion of $sp\ell(2,1)$ in [11] by $H = J_3$, $E^{\pm} = J_1 \pm iJ_2$, $\sqrt{2}F^{+} = Q_1^{\dagger}$, $\sqrt{2}F^{-} = -Q_2^{\dagger}$, $\sqrt{2}\,\overline{F}^{+} = -Q_2$, $\sqrt{2}\,\overline{F}^{-} = Q_1$ when the upper sign in (7) is used.)

Norms of states can be computed; we find

$$< 1|1 >=< I| \left\{ Q_1, Q_1^{\dagger} \right\} |I >= (\zeta \mp 2m) < I|I > \tag{36a}$$

$$< 2|2 >= (\zeta \pm 2m) < I|I > \tag{36b}$$

$$< F|F >= (\zeta \mp 2j)(\zeta \pm 2j \pm 2) < I|I > \tag{36c}$$

as

$$(J_1 \pm iJ_2)\,|j,m >= \left[(j \mp m)(j \pm m + 1)\right]^{1/2} |j, m \pm 1 > . \tag{37}$$

For the norm of these states to be positive definite, we must have

$$2j < \zeta \tag{38}$$

so that ζ forms an upper bound on j. This is similar to the way in which the central charge forms an upper bound on the magnitude of the momentum in the supersymmetric extension of the Poincaré group in $4+0$ dimensions and $5+0$ dimensions [3,12]. By way of contrast, in the supersymmetric extension of the Poincaré group in $3+1$ dimensions or $4+1$ dimensions, the central charge forms a lower bound on the magnitude of the momentum [2,13-15,12].

4.2 AdS_2

Representations of the $N = 2$ superalgebra of (22) and (19) (and consequently of (16)) can be worked out applying techniques used in [8] for the $N = 1$ superalgebra. The $N = 2$

superalgebra is also considered in [9] and [16]; in the latter there is no analogue of the operator α which is essential for the Jacobi identities.

Working directly with the operators of (16) (using the upper sign), the subalgebra of (16a) has a Casimir

$$\not{C}_0 = \delta^2 - \tfrac{1}{2}\{\pi, \kappa\} \tag{39}$$

while the full superalgebra of (16) has the Casimir

$$\not{C} = \not{C}_0 + A - \tfrac{1}{4}\alpha^2 \tag{40}$$

where

$$A = A_1 + A_2 \tag{41}$$

with

$$A_i = -\tfrac{1}{2}[q_i, s_i] \quad (i = 1, 2). \tag{42}$$

Since

$$4A_1^2 - 2A_1 = \not{C}_0 = 4A_2^2 - 2A_2 \tag{43}$$

we see that

$$\not{C} = 2\left(A_1^2 + A_2^2\right) - \tfrac{1}{4}\alpha^2. \tag{44}$$

A general discussion of the $S0(2,1)$ group of (16a) appears in [17]. The $S0(2)$ subgroup of $S0(2,1)$ has a Casimir

$$R = \tfrac{1}{2}(\kappa - \pi). \tag{45}$$

We can now classify states $|\psi>$ by eigenvalues of R, $\not{C}_0$, $\not{C}$ and α (taken to be ρ, γ_0, γ and a respectively). From (43) we see that

$$A_i|\psi>= A_i|\rho, \gamma_0, \gamma, a>= \frac{1+\epsilon_i\sqrt{1+4\gamma_0}}{4}\,|\rho, \gamma_0, \gamma, a> \tag{46}$$

where $\epsilon_i = \pm 1$; thus by (41)

$$\gamma = \gamma_0 = \tfrac{1}{4}\left(2 + (\epsilon_1 + \epsilon_2)\sqrt{1+4\gamma_0} - a^2\right). \tag{47}$$

Ladder operators for $S0(2,1)$ are given by

$$B_\pm = \tfrac{1}{2}(\kappa + \pi) \mp \delta = B_\mp^\dagger; \tag{48}$$

in addition, there are Fermionic ladder operators

$$F_\pm^i = q^i \pm i s_i. \tag{49}$$

These operators are related by

$$\not{C}_0 = R^2 - \tfrac{1}{2}\left\{B_-, B_+\right\} \tag{50a}$$

$$[R, B_\pm] = \pm B_\pm \tag{50b}$$

$$\left[R, F_\pm^i\right] = \pm\tfrac{1}{2}F_\pm^i \tag{50c}$$

$$\left\{F_+^i, F_-^j\right\} = -2i\delta_{ij}R - \epsilon_{ij}\alpha \tag{50d}$$

$$\left\{F_\pm^i, F_\pm^j\right\} = \tfrac{i}{2}\delta_{ij}B_\pm \tag{50e}$$

$$\left[F_\pm^i, \alpha\right] = -i\epsilon_{ij}F_\pm^j \tag{50f}$$

$$A_i = \tfrac{i}{4}\left[F_-^i, F_+^i\right] \tag{50g}$$

$$B_\pm B_\mp = R^2 \mp R - \not{C}_0 \tag{50h}$$

$$F^i_\pm F^i_\mp = -2iR \mp 2iA \tag{50i}$$

$$= -2iR \mp 2i \left(\not{C} - \not{C}_0 + \tfrac{1}{4}\alpha^2 \right)$$

$$\left[B_\pm , F^i_\pm \right] = 0 \tag{50j}$$

$$\left[B_\pm , F^i_\mp \right] = \pm F^i_\pm . \tag{50k}$$

If now we define

$$|\psi\pm >= B_\pm |\psi > \tag{51a}$$

$$|\psi i\pm >= F^i_\pm |\psi > \tag{51b}$$

then it follows that

$$\not{C} |\psi\pm >= \gamma |\psi\pm > \tag{52a}$$

$$\not{C}_0 |\psi\pm >= \gamma_0 |\psi\pm > \tag{52b}$$

$$R|\psi\pm >= (\rho \pm 1)|\psi\pm > \tag{52c}$$

$$\alpha|\psi\pm >= a|\psi\pm > \tag{52d}$$

as well as

$$\not{C} |\psi i\pm >= \gamma |\psi i\pm > \tag{53a}$$

$$R|\psi i\pm >= (\rho \pm \tfrac{1}{2})|\psi i\pm > \tag{53b}$$

$$\alpha|\psi i\pm >= \left(+i\epsilon_{ij} + a\delta_{ij} \right) |\psi j\pm > \tag{53c}$$

$$\left(A + \not{C}_0 \right) |\psi i\pm >= \left(\not{C}_0 + \tfrac{1}{4}\alpha^2 \right) |\psi i\pm > \tag{53d}$$

$$= \left[\left(+\tfrac{1}{4} + \gamma + \tfrac{1}{4}a^2\right)\delta_{ij} + \tfrac{i}{2}a\epsilon_{ij}\right] |\psi j\pm > .$$

(The $S0(2,1)$ invariant $A + \not{C}_0$ is easier to work with than $\not{C}_0$ itself as $\left[\not{C}_0, F^i_\pm\right]$ is non-trivial while $\left[A + \not{C}_0,\ F^i_\pm\right]$ is easily evaluated.) One can now diagonalzie the 2×2 matrices on the right side of (53c) and (53d).

Using (50h) and (50i), it follows that

$$< \psi \pm |\psi\pm >= \left(\rho^2 \pm \rho - \gamma_0\right) < \psi|\psi > \tag{54a}$$

and

$$< \psi i \pm |\psi i\pm >= \left[-2\rho \mp 2\left(\gamma - \gamma_0 + \tfrac{1}{4}\alpha^2\right)\right] < \psi|\psi > . \tag{54b}$$

The forms for γ_0 and ρ are [17]

$$\gamma_0 = \Phi(\Phi + 1) \tag{55a}$$

$$\rho = E_0 + n \quad \left(-\tfrac{1}{2} \leq E_0 < \tfrac{1}{2}, n \text{ integer}\right) \tag{55b}$$

with the permitted values of Φ and E_0 falling into four distinct classes. (In [8] the allowed values of Φ and E_0 are restricted by an additional $0(3)$ symmetry present in the physical system being considered.)

5 Models in Two Dimensions

It is possible to write down component field models in two dimensions that are invariant under the supersymmetry transformations associated with the superalgebras of (7), (19) and (21). They bear a certain resemblance to the hyperspherical models considered in [18-30].

5.1 S_2

It is possible to present a model invariant under transformations associated with the algebra of eq. (7) (with the upper sign). We consider the action

$$S = \int \frac{dA}{R^2}\Bigg[\Big(\tfrac{1}{2}\Psi^\dagger\left(\vec{\tau}\cdot\vec{L}+x\right)\Psi - \Phi^*\left(L^2+x(1-x)\right)\Phi$$

$$-\tfrac{1}{4}F^*F\Big) + \lambda_N\Big(2(1-2x)\Phi^*\Phi$$

$$-(F^*\Phi+F\Phi^*) - \Psi^\dagger\Psi\Big)^N\Bigg]. \tag{56}$$

In (56), Φ and F are complex scalars, and Ψ is a Dirac spinor, defined on the surface of a sphere of radius R in three dimensions. The angular momentum operator is $\vec{L} = -i\vec{r}\times\vec{\nabla}$ and x and λ_N are arbitrary real parameters.

By using the identities of eqs. (A.1 - A.4), one can verify that for arbitrary N, (56) is invariant under both the supersymmetry transformation

$$\delta\Phi = \xi^\dagger\Psi$$

$$\delta\Psi = 2\left(\vec{\tau}\cdot\vec{L}+1-x\right)\Phi\xi - F\xi \tag{57}$$

$$\delta F = -2\xi^\dagger\left(\vec{\tau}\cdot\vec{L}+x\right)\Psi$$

and the special transformations

$$\delta\Psi = \lambda i\left(1+2\vec{\tau}\cdot\vec{L}\right)\Psi$$

$$\delta\Phi = \lambda i\left(2(1-x)\Phi - F\right) \tag{58}$$

$$\delta F = \lambda i\left[-4\left(L^2+x(1-x)\right)\Phi + 2xF\right]$$

where ξ is a constant Grassmann spinor and λ is a constant. These transformations are generated by $\exp\left[\xi^\dagger R - R^\dagger \xi + i\lambda Z + i\vec{\omega}\cdot\vec{J}\right]$ using the commutators

$$\left[\xi^\dagger R, \Phi\right] = \xi^\dagger \Psi \quad , \qquad \left[\xi^\dagger R, \Psi\right] = \left[2(\vec{\tau}\cdot\vec{L} + 1 - x)\Phi - F\right]\xi$$

$$\left[\xi^\dagger R, F\right] = -2\xi^\dagger(\vec{\tau}\cdot\vec{L} + x)\Psi \tag{59}$$

$$[\lambda Z, \Phi] = -\lambda\left[2(1-x)\Phi - F\right], \qquad [\lambda Z, \Psi] = -\lambda\left[1 + 2\vec{\tau}\cdot\vec{L}\right]\Psi$$

$$[\lambda Z, F] = -\lambda\left[-4\left(\vec{L}^2 + x(1-x)\right)\Phi + 2xF\right]$$

$$\left[\vec{\omega}\cdot\vec{J}, \Phi\right] = -\vec{\omega}\cdot\vec{L}\Phi, \qquad \left[\vec{\omega}\cdot\vec{J}, \Psi\right] = -\vec{\omega}\cdot\left(\vec{L} + \tfrac{1}{2}\vec{\tau}\right)\Psi$$

$$\left[\vec{\omega}\cdot\vec{J}, F\right] = -\vec{\omega}\cdot\vec{L}F.$$

(All other commutators vanish.) Jacobi identities involving (7) (upper sign) and (59) are satisfied.

Other models on S_2 also exist which possess a Fermionic symmetry. For example let us consider

$$S = \int \frac{dA}{R^2}\left[i\Psi^\dagger\vec{\tau}\cdot\vec{r}\left(\vec{\tau}\cdot\vec{L} + 1\right)\Psi + \Phi^*\vec{L}^2\Phi\right]. \tag{60}$$

This differs in form from the action considered in [21] by the factor of i in the first term (needed to ensure Hermiticity as $\vec{\tau}\cdot\vec{r}\left(\vec{\tau}\cdot\vec{L} + 1\right) = -\left(\vec{\tau}\cdot\vec{L} + 1\right)\vec{\tau}\cdot\vec{r}$). The action of eq. (60) possess the superinvariance

$$\delta\Phi = R^2\xi^\dagger\Psi \qquad \delta\Psi = -i\vec{\tau}\cdot\vec{r}\vec{\tau}\cdot\vec{L}\Phi\xi \tag{61}$$

where ξ is again a Grassmann spinor. This symmetry however does not appear to be consistent with the superalgebra of (7).

As a second model, let us take

$$S=\int\frac{dA}{R^2}\left[\tfrac{1}{2}\Psi^\dagger\left(\vec{\tau}\cdot\vec{L}+x\right)\Psi+\Phi^*\left(\vec{L}^2+x(3-x)-2\right)\Phi\right]. \tag{62}$$

This is invariant under

$$\delta\Phi=\xi^\dagger\vec{\tau}\cdot\vec{r}\Psi \qquad \delta\Psi=-2\left(\vec{\tau}\cdot\vec{L}+3-x\right)\vec{\tau}\cdot\vec{r}\Phi\xi \tag{63}$$

as $[\vec{\tau}\cdot\vec{r},L^2]=-2\vec{\tau}\cdot\vec{r}\left(\vec{\tau}\cdot\vec{L}+1\right)$, but once again this symmetry does not appear to be consistent with (7). We note that $\vec{\tau}\cdot\vec{r}$ has many of the properties of γ_5 in Euclidean space [31,32].

5.2 AdS_2

Models associated with the superalgebras of (19) and (21) can also be devised. The model

$$S=\int\frac{dA}{R^2}\left[\tilde{\Psi}\left(\Sigma^{ab}L_{ab}+\chi\right)\Psi+\Phi\left(\tfrac{1}{2}L^{ab}L_{ab}\right.\right. \tag{64}$$

$$\left.\left.+\chi(1+\chi)\Phi\right)-F^2+\lambda_N\left((1+2\chi)\Phi^2+2\Phi F+\tilde{\Psi}\Psi\right)^N\right]$$

is invariant under

$$\delta\Psi=\left[\left(\Sigma^{ab}L_{ab}-(1+\chi)\right)\Phi-F\right]\xi$$

$$\delta\Phi=\tilde{\xi}\Psi\ \ ,\delta F=-\tilde{\xi}\left(\Sigma^{ab}L_{ab}+\chi\right)\Psi \tag{65}$$

if $L_{ab}=-x_a\partial_b+x_b\partial_a$. (Note that $\left(\Sigma^{ab}L_{ab}\right)^2=-\frac{1}{2}L^{ab}L_{ab}+\Sigma^{ab}L_{ab}$.) The superalgebra of (19) is consistent with the transformations of (65).

In conjunction with the superalgebra of (21), (an $N = 2$ supersymmetry algebra), one has an invariant model whose action is

$$S = \int \frac{dA}{R^2} \left[\overline{\Psi} \left(\Sigma^{ab} L_{ab} + \chi \right) \Psi + \Phi^* \left(\tfrac{1}{2} L^{ab} L_{ab} \right. \right.$$

$$\left. \left. + \chi(1+\chi) \right) \Phi - F^* F \right] \tag{66}$$

with now

$$\delta\Psi = \left[\left(\Sigma^{ab} L_{ab} - (1+\chi) \right) \Phi - F \right] \xi \tag{67}$$

$$\delta\Phi = \overline{\xi}\Psi \quad , \delta F = -\overline{\xi} \left(\Sigma^{ab} L_{ab} + \chi \right) \Psi.$$

In (64) Ψ is a Majorana spinor and Φ and F are real scalars; in (64) Ψ is a Dirac spinor and Φ and F are complex spinors.

6 Superspace for Two Dimensions

We can realize the two dimensional superalgebras of eqs. (7) and (19) in superspace. Superspace models invariant under (19) can be formulated.

6.1 S_2

In conjunction with the two superalgebras of (7), we consider a superspace with coordinates x^a, θ_i and $\theta_i^\dagger$ where θ_i is a two component Dirac Grassmann spinor. In addition, we employ an auxiliary constant β. The two superalgebras of (7) have a representation [33]

$$Q = (\vec{\tau} \cdot \vec{x} + \beta) \frac{\partial}{\partial \theta^\dagger} \pm \left(\frac{\partial}{\partial \beta} - \vec{\tau} \cdot \vec{\nabla} \right) \theta$$

$$Q^\dagger = \frac{\partial}{\partial\theta}\left(\vec{\tau}\cdot\vec{x}+\beta\right) \mp \theta^\dagger\left(\frac{\partial}{\partial\beta} - \vec{\tau}\cdot\vec{\nabla}\right) \tag{68}$$

$$J^a = \tfrac{1}{2}\left[\frac{\partial}{\partial\theta}\tau^a\theta + \theta^\dagger\tau^a\frac{\partial}{\partial\theta^\dagger}\right] + \left(-i\vec{x}\times\vec{\nabla}\right)^a$$

$$Z = \pm\left(\theta^\dagger\frac{\partial}{\partial\theta^\dagger} - \theta\frac{\partial}{\partial\theta}\right).$$

Two quantities that commute with Q are $\vec{x}^2 - \beta^2 \pm 2\theta^\dagger\theta$ and $\theta^\dagger\frac{\partial}{\partial\theta^\dagger} + \theta\frac{\partial}{\partial\theta} + \vec{x}\cdot\vec{\nabla} + \beta\frac{\partial}{\partial\beta}$. The changes induced by Q and $Q^\dagger$ on x^a and θ are

$$\delta x^a = \left[\xi^\dagger Q - Q^\dagger\xi, x^a\right] = \pm\left(\theta^\dagger\tau^a\xi - \xi^\dagger\tau^a\theta\right)$$

$$\delta\theta = \left[\xi^\dagger Q - Q^\dagger\xi, \theta\right] = \left(\vec{\tau}\cdot\vec{x}+\beta\right)\xi. \tag{69}$$

The physical interpretation of β is not clear. Also, it is not apparent how to construct a superfield containing component fields (such as appear in (56)) that constitute an irreducible representation of the supersymmetry transformation induced by (68). This may be due to an inability to find a Grassmann operator that anticommutes with Q and $Q^\dagger$ as represented in (68).

6.2 AdS_2

In conjunction with the superalgebra of (19) we have a superspace composed of Bosonic coordinates x^a and Fermionic coordinates θ_i where θ_i is a two component Majorana spinor. A suitable representation of this superalgebra in superspace is provided by

$$J^{ab} = \frac{\partial}{\partial\theta}\Sigma^{ab}\theta - \left(x^a\partial^b - x^b\partial^a\right)$$

$$Q = \gamma^a\partial_a\theta + \gamma^a x_a\frac{\partial}{\partial\tilde{\theta}} \tag{70}$$

$$\tilde{Q} = -\tilde{\theta}\gamma^a\partial_a + \frac{\partial}{\partial\theta}\gamma^a x_a.$$

No analogue of the variable β appearing in (68) is needed. Two quantities that commute with Q appearing in (70) are

$$R^2 = x^a x_a - \tilde{\theta}\theta \tag{71}$$

and

$$\Delta = x^a\partial_a + \theta_i\frac{\partial}{\partial\theta_i} = x^a\partial_a + \tilde{\theta}_i\frac{\partial}{\partial\tilde{\theta}_i}. \tag{72}$$

We also can define

$$\begin{aligned} D &= -\gamma^a\partial_a\theta + \gamma^a x_a\frac{\partial}{\partial\tilde{\theta}} \\ \tilde{D} &= \tilde{\theta}\gamma^a\partial_a + \frac{\partial}{\partial\theta}\gamma^a x_a. \end{aligned} \tag{73}$$

Several useful relations are

$$\begin{gathered} [D_j, \Delta] = 0 \\ \left\{Q_i, \tilde{D}_j\right\} = -2\overline{\Delta}\delta_{ij} \\ \left(\overline{\Delta} \equiv x^a\partial_a + \tfrac{3}{2}\theta_i\frac{\partial}{\partial\theta_i}\right) \\ \left[Q_i, \overline{\Delta}\right] = \tfrac{1}{2}D_i \\ \left[Q_i, D\tilde{D}\right] = 2\left[D_i, \overline{\Delta}\right] = Q_i. \end{gathered} \tag{74}$$

From (72), we see that one can define a supersymmetric invariant condition

$$\Delta\Phi = \omega\Phi \tag{75}$$

on a scalar superfield Φ . This is analogous to the homogeneity condition used in dS_4 space in [33]. A suitable invariant action can be taken to be

$$S_1 = \int d^3x d^2\theta \delta(R^2 - a^2)\Phi(\tilde{D}D + \rho)\Phi. \tag{76}$$

If now

$$\Phi = \phi + \tilde{\lambda}\theta + F\tilde{\theta}\theta \tag{77}$$

then by (75)

$$(x \cdot \partial - \omega)\phi = (x \cdot \partial + 1 - \omega)\lambda = (x \cdot +2 - \omega)F = 0. \tag{78}$$

Noting that

$$\begin{aligned} \tilde{D}D = & -x^2 \frac{\partial}{\partial \theta}\frac{\partial}{\partial \tilde{\theta}} + \frac{1}{2x^2}\left[L^{ab}L_{ab} + 2(x\cdot\partial)^2 + 2(x\cdot\partial)\right]\tilde{\theta}\theta \\ & +2x\cdot\partial + 2\tilde{\theta}\left(-x\cdot\partial + \Sigma^{ab}L_{ab}\right)\frac{\partial}{\partial\tilde{\theta}} - 3\tilde{\theta}\frac{\partial}{\partial\tilde{\theta}} \end{aligned} \tag{79}$$

and

$$\delta\left(R^2 - a^2\right) = \delta\left(x^2 - a^2\right) - \tilde{\theta}\theta\delta'\left(x^2 - a^2\right)$$

we see that the component form of (76) is

$$\begin{aligned} S_1 = \int d^3x \Big\{ & \delta\left(x^2 - a^2\right)\left[-\tilde{\lambda}\left(\Sigma^{ab}L_{ab} + \frac{\rho - 3}{2}\right)\lambda \right. \\ & + \frac{1}{2x^2}\phi\left(L^{ab}L_{ab} + 2\omega(1-\omega)\right)\phi - 2x^2F \\ & \left. +2(\rho - 1)\phi F\right] + \delta'\left(x^2 - a^2\right)\left[2x^2\phi F \right. \\ & \left. -(\rho + 2\omega)\phi^2\right]\Big\}. \end{aligned} \tag{80}$$

Upon integrating over $\sqrt{x^2}$, we find that the action on the AdS_2 surface is

$$S_1 = \int d^2A a^2 \left[-\tilde{\lambda}\left(\Sigma^{ab}L_{ab} + \frac{\rho - 3}{2}\right)\lambda\right. \tag{81}$$

$$+\frac{1}{2a^3}\phi\left(L^{ab}L_{ab}+2\omega(1+5\omega+\rho)+\rho\right)\phi$$

$$-2a^2F^2+(2\rho-3-2\omega)\phi F\Big].$$

If $\delta\Phi=\left[\tilde{\xi}Q,\Phi\right]$, then it follows that

$$\delta\phi=i\tilde{\xi}\gamma\cdot x\lambda$$

$$\delta\tilde{\lambda}=i\tilde{\xi}\gamma\cdot(\partial\phi+2xF) \tag{82}$$

$$\delta F=-\tfrac{i}{2}\tilde{\xi}\gamma\cdot\partial\lambda.$$

There is no immediate connection between the actions of (64) and (81), although the changes of (82) can be identified with those of (65) provided $i\gamma\cdot x\lambda\to\Psi$, $\omega\to -1-\chi$ and $2x^2F\to -F$ in (82).

In place of (76) one could also consider the supersymmetric invariant actions

$$S_2=\int d^3xd^2\theta\delta\left(R^2-a^2\right)\Phi(\tilde{Q}Q+\rho)\Phi \tag{83a}$$

$$S_3=\int d^3xd^2\theta\delta\left(R^2-a^2\right)\left[(\tilde{D}\Phi)(D\Phi)+\rho\Phi^2\right] \tag{83b}$$

$$S_4=\int d^3xd^2\theta\delta\left(R^2-a^2\right)\left[(\tilde{Q}\Phi)(Q\Phi)+\rho\Phi^2\right] \tag{83c}$$

as well as supersymmetric invariant interactions

$$S_I=\lambda_N\int d^3xd^2\theta\delta\left(R^2-a^2\right)\Phi^N. \tag{84}$$

(Actually, (83a) and (83c) are identical as $[Q_i,R^2]=0$.)

Establishing supersymmetric invariance of $S_1\cdots S_4$ in (76) and (83) is not easily done if one works directly in terms of component fields as in (81). However one can argue as follows to establish this invariance. The expansion $(\tilde{D}D + \rho)\Phi$ is itself a scalar superfield given by

$$P + \tilde{S}\theta + R\tilde{\theta}\theta = \left(-2x^2 F + 2\omega\phi\right) + \tilde{\theta}\left(2\Sigma^{ab}L_{ab} - 3\right)\lambda$$

$$+\tilde{\theta}\theta\left[-2(1+\omega)\phi + \frac{1}{2x^2}\left(L^{ab}L_{ab} + 2\omega(1+\omega)\right)\phi\right] \tag{85}$$

and hence the change in the product of two scalar fields is given by

$$\left[\tilde{\xi}Q, \left(P_1 + \tilde{\theta}S_1 + R_1\tilde{\theta}\theta\right)\left(P_2 + \tilde{\theta}S_2 + R_2\tilde{\theta}\theta\right)\right]$$

$$= \Big[\tilde{\xi}Q, P_1P_2 + \left(P_1\tilde{S}_2 + P_2\tilde{S}_1\right)\theta + (P_1R_2 + P_2R_1$$

$$-\tfrac{1}{2}\tilde{S}_1S_2\Big)\tilde{\theta}\theta\Big]$$

$$= -\tfrac{1}{2}\partial_a\left[\tilde{\xi}\gamma^a\left(P_1S_2 + P_2S_1\right)\right]\tilde{\theta}\theta + O(\theta).$$

Since this is the divergence of a current at order $\tilde{\theta}\theta$, S_1, S_2, S_3 and S_4 are all supersymmetric invariant actions.

7 Discussion

In this paper we have presented the simplest superalgebras associated with spaces of constant curvature in two, three and four dimensions. In this way, some of the superalgebras considered in [35] are exhibited explicitly. In addition, some novel superalgebras (eg, that of (13)) have been noted whcih do not fall into the categories considered by Nahm [37] or Pilch et al. [38].

We have also considered the two dimensional models in more detail. In particular, we have examined their representations. One peculiar result occurs in the case of the superalgebra of eq. (7) associated with S_2, namely the requirement that states have positive definite norm restricts the angular momentum to be less than a value given in terms of the eigenvalues of the internal symmetry generator (cf eq. (38)). In addition, component field models associated with superalgebras related to AdS_2 and S_2 have been devised, and superfield models invariant under the AdS_2 superalgebra are given.

Clearly more work remains to be done. Formulating models in three and four dimensions is a high priority. Considering spaces of constant curvature in dimensions higher than four also merits attention. It may also be possible to relate the algebra of (13) to non-commutative geometry [39]. These questions currently are under consideration.

8 Appendix

8.1 Three Dimensions

In $3+0$ dimensions, the Dirac matrices can be identified with the Pauli spin matrices τ^a_{ij}. These satisfy

$$\tau^a \tau^b = \delta^{ab} + i\epsilon^{abc}\tau^c \tag{A.1}$$

$$\tau^a_{ij}\tau^a_{k\ell} = 2\delta_{i\ell}\delta_{kj} - \delta_{ij}\delta_{k\ell} \tag{A.2}$$

$$\tau^a_{ij}\delta_{k\ell} + \tau^a_{k\ell}\delta_{ij} = \tau^a_{i\ell}\delta_{kj} + \tau^a_{kj}\delta_{i\ell} \tag{A.3}$$

$$\epsilon^{abc}\tau^b_{ij}\tau^c_{k\ell} = i\left(\tau^a_{i\ell}\delta_{kj} - \tau^a_{kj}\delta_{i\ell}\right). \tag{A.4}$$

Charge conjugation of a spinor ψ is given by $\psi_c = C(\psi^\dagger)^T$ where

$$C^{-1}\tau^\mu C = -(\tau^\mu)^T; \tag{A.5}$$

C is taken to be

$$C = \tau^2 = C^\dagger = C^{-1} = -C^T = -C^*. \tag{A.6}$$

In $2+1$ dimensions, we use the metric $\eta_{ab} = \mathrm{diag}(+,-,+)$ and choose $\gamma_1 = i\tau_1$, $\gamma_2 = \tau_2$, $\gamma_3 = i\tau_3$ so that

$$\gamma_a \gamma_b = -\eta_{ab} - i\epsilon_{abc}\gamma^c \tag{A.7}$$

as $\epsilon^{123} = +1$. We take also

$$\Sigma_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right] = -\frac{i}{2}\epsilon_{abc}\gamma^c. \tag{A.8}$$

Since

$$\gamma_2 \Sigma_{ab} \gamma_2 = -\Sigma_{ab}^T = -\Sigma_{ab}^\dagger \tag{A.9}$$

both $\overline{Q}Q$ and $\tilde{Q}Q$ are invariant under the transformation

$$Q \to \exp\left(-\frac{1}{2}\omega_{ab}\Sigma^{ab}\right) Q \tag{A.10}$$

where

$$\overline{Q} = Q^\dagger \gamma_2 \quad , \tilde{Q} = Q^T \gamma_2. \tag{A.11a, b}$$

The matrix C is defined by (A.5) and (A.6) in both $3dM$ and $3dE$; in $3dM$ $\psi_C = C\overline{\psi}^T$.

If in $2+1$ dimensions, θ, ξ and χ are all Majorana (viz. $\theta = \theta_c$, $\xi = \xi_c$, $\chi = \chi_c$) then we have

$$\left(\tilde{\xi}\chi\right) = \left(\tilde{\chi}\xi\right) = \left(\tilde{\xi}\chi\right)^\dagger \tag{A.12}$$

$$\left(\tilde{\xi}\gamma^a\chi\right) = -\left(\tilde{\chi}\gamma^a\xi\right) \tag{A.13}$$

$$\left(\tilde{\xi}\theta\right)\left(\tilde{\theta}\chi\right) = -\frac{1}{2}\left(\tilde{\xi}\chi\right)\left(\tilde{\theta}\theta\right). \tag{A.14}$$

In $3+0$ dimensions, $(\psi_c)_c = -\psi$ so one cannot impose the Majorana condition; spinors can be Dirac, or alternatively, one can have a pair of "symplectic Majorana" spinors Q_i satisfying

$$Q_i = \epsilon_{ij}(Q_c)_j \tag{A.15}$$

where $\epsilon_{ij} = -\epsilon_{ji}$, $\epsilon_{12} = 1$.

8.2 Four Dimensions

In $4+0$ dimensions, we choose the following representation for the Dirac matrices

$$\gamma^i = \begin{pmatrix} 0 & i\tau^i \\ -i\tau^i & 0 \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.16}$$

$$\Sigma^{\mu\nu} = -\tfrac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \left[\Sigma^{\alpha\beta}, \gamma^\gamma\right] = \delta^{\alpha\gamma}\gamma^\beta - \delta^{\beta\gamma}\gamma^\alpha, \quad \left[\Sigma^{\mu\nu}, \Sigma^{\lambda\sigma}\right] = \delta^{\mu\lambda}\Sigma^{\nu\sigma} + \cdots \tag{A.17}$$

Contracting the expansion

$$\Sigma^{\mu\nu}_{ij}\Sigma^{\mu\nu}_{k\ell} = a_{i\ell}\delta_{kj} + a^5_{i\ell}\gamma^5_{kj} + a^\mu_{i\ell}\gamma^\mu_{kj} + a^{\mu5}_{i\ell}\left(\gamma^\mu\gamma^5\right)_{kj}$$

$$+a^{\mu\nu}_{i\ell}\Sigma^{\mu\nu}_{kj} \tag{A.18}$$

with δ_{kj}, γ^5_{kj}, γ^μ_{kj}, $(\gamma^\mu\gamma^5)_{kj}$ and $\Sigma^{\mu\nu}_{kj}$ in turn leads to

$$\Sigma^{\mu\nu}_{ij}\Sigma^{\mu\nu}_{k\ell} = -\tfrac{1}{2}\Sigma^{\mu\nu}_{i\ell}\Sigma^{\mu\nu}_{kj} - \tfrac{3}{4}\left(\delta_{i\ell}\delta_{kj} + \gamma^5_{i\ell}\gamma^5_{kj}\right) \tag{A.19}$$

so that

$$\Sigma^{\mu\nu}_{ij}\Sigma^{\mu\nu}_{k\ell}+\Sigma^{\mu\nu}_{i\ell}\Sigma^{\mu\nu}_{kj}=-\tfrac{1}{2}\left[(\delta_{ij}\delta_{k\ell}+\delta_{i\ell}\delta_{kj})+\left(\gamma^5_{ij}\gamma^5_{k\ell}+\gamma^5_{i\ell}\gamma^5_{kj}\right)\right]. \tag{A.20}$$

Similarly, one has

$$\delta_{ij}\delta_{k\ell}=\tfrac{1}{4}\left[\delta_{i\ell}\delta_{kj}+\gamma^5_{i\ell}\gamma^5_{kj}+\gamma^\mu_{i\ell}\gamma^\mu_{kj}-(\gamma^\mu\gamma^5)_{i\ell}(\gamma^\mu\gamma^5)_{kj}\right.$$
$$\left.-2\Sigma^{\mu\nu}_{i\ell}\Sigma^{\mu\nu}_{kj}\right]. \tag{A.21}$$

Together, (A.20) and (A.21) give

$$2\left(\delta_{ij}\delta_{k\ell}+\delta_{i\ell}\delta_{kj}\right)=\left(\gamma^a_{ij}\gamma^a_{k\ell}+\gamma^a_{i\ell}\gamma^a_{kj}\right)-\left[\left(\gamma^a\gamma^5\right)_{ij}\left(\gamma^a\gamma^5\right)_{k\ell}\right.$$
$$\left.+\left(\gamma^a\gamma^5\right)_{i\ell}\left(\gamma^a\gamma^5\right)_{kj}\right]+2\left[\left(\gamma^5\right)_{ij}\left(\gamma^5\right)_{k\ell}+\left(\gamma^5\right)_{i\ell}\left(\gamma^5\right)_{kj}\right]. \tag{A.22}$$

For charge conjugation, we now take

$$C\gamma^\mu C^{-1}=-(\gamma^\mu)^T \tag{A.23}$$

and with $C=\gamma^0\gamma^2$,

$$C=-C^T=-C^\dagger=-C^{-1}=C^*. \tag{A.24}$$

If $\psi_C=C(\psi^\dagger)^T$, then $(\psi_C)_C=-\psi$ so a spinor cannot be Majorana; it can be Dirac, or we can have a pair of symplectic Majorana spinors.

In $3+1$ dimensions, where $g^{\mu\nu}=\mathrm{diag}(+---)$, we choose

$$\gamma^i=\begin{pmatrix}0 & \tau^i\\ -\tau^i & 0\end{pmatrix}\quad \gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\quad \gamma^5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{A.25}$$

The charge conjugation matrix C is taken to be $-i\gamma^0\gamma^2$ and so that if

$$\overline{\psi}=\psi^\dagger\gamma^0 \quad ,\psi_C=C(\overline{\psi})^T \tag{A.25}$$

then it is apparent that the Majorana condition can be imposed.

One can also use 2-component notations for spinors in $3+1$ dimensions, using the conventions of [35]. However, we find it convenient to avoid distinguishing between upper and lower case indices and to not use the dot notation for indices; rather we choose to strictly employ matrix notation.

We first define

$$\sigma^\mu = (1, \vec{\tau}) \quad , \quad \overline{\sigma}^\mu = (1, -\vec{\tau})$$

$$\sigma^{\mu\nu} = -\tfrac{1}{4}\left(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu\right), \qquad \overline{\sigma}^{\mu\nu} = -\tfrac{1}{4}\left(\overline{\sigma}^\mu \sigma^\nu - \overline{\sigma}^\nu \sigma^\mu\right).$$

These satisfy the relations

$$\sigma^{\mu\dagger} = \sigma^\mu \quad , \quad \overline{\sigma}^{\mu\dagger} = \overline{\sigma}^\mu \ , \quad \sigma^{\mu\nu\dagger} = -\overline{\sigma}^{\mu\nu}$$

$$\sigma_2 \sigma^\mu \sigma_2 = \overline{\sigma}^{\mu T} \qquad \sigma_2 \sigma^{\mu\nu} \sigma_2 = -\sigma^{\mu\nu T}$$

$$\left[\sigma^{\mu\nu}, \sigma^{\lambda\sigma}\right] = g^{\mu\lambda}\sigma^{\nu\sigma} - g^{\mu\sigma}\sigma^{\nu\lambda} + g^{\nu\sigma}\sigma^{\mu\lambda} - g^{\nu\lambda}\sigma^{\mu\sigma}$$

$$\sigma^{\mu\nu}\sigma^\lambda = \tfrac{1}{2}\left(g^{\mu\lambda}\sigma^\nu - g^{\nu\lambda}\sigma^\mu + i\epsilon^{\mu\nu\lambda\rho}\sigma_\rho\right)$$

$$\overline{\sigma}^\mu \sigma^{\nu\lambda} = -\tfrac{1}{2}\left(g^{\mu\nu}\overline{\sigma}^\lambda - g^{\mu\lambda}\overline{\sigma}^\nu + i\epsilon^{\mu\nu\lambda\rho}\overline{\sigma}_\rho\right) \quad \left(\epsilon^{1230} = +1\right)$$

$$\left(\sigma^{\mu\nu}\right)_{ij}\left(\sigma_{\mu\nu}\right)_{k\ell} = -2\delta_{i\ell}\delta_{kj} + \delta_{ij}\delta_{k\ell} \ , \quad \left(\sigma^{\mu\nu}\right)_{ij}\left(\overline{\sigma}_{\mu\nu}\right)_{k\ell} = 0 \quad \left(\sigma^\mu\right)_{ij}\left(\overline{\sigma}_\mu\right)_{k\ell} = 2\delta_{i\ell}\delta_{kj}.$$

Consider now 2-component spinors ψ and χ and define

$$\tilde{\psi} = \psi^T \sigma_2 \quad \tilde{\chi} = \chi^T \sigma_2 \ .$$

It follows from the above relations that if

$$\psi \rightarrow e^{\omega_{\mu\nu}\sigma^{\mu\nu}}\psi \quad \chi \rightarrow e^{\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}}\chi$$

then

$$\tilde{\psi} \to \tilde{\psi} e^{-\omega_{\mu\nu}\sigma^{\mu\nu}} \quad \tilde{\chi} \to \tilde{\chi} e^{-\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}}$$

$$\psi \to \psi^{\dagger} e^{-\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}} \quad \chi^{\dagger} \to \chi^{\dagger} e^{-\omega_{\mu\nu}\sigma^{\mu\nu}}$$

$$\left(\sigma_2 \psi^{\dagger T}\right) \to e^{\omega_{\mu\nu}\overline{\sigma}^{\mu\nu}} \left(\sigma_2 \psi^{\dagger T}\right) \quad \left(\sigma_2 \chi^{\dagger T}\right) \to e^{\omega_{\mu\nu}\sigma^{\mu\nu}} \left(\sigma_2 \chi^{\dagger T}\right).$$

It is apparent then that the following structures are Lorentz invariant:

$$\tilde{\psi}\psi\,, \quad \tilde{\chi}\chi\,, \quad \psi^{\dagger}\overline{\sigma}_{\mu}\psi\,, \quad \chi^{\dagger}\sigma_{\mu}\chi\,, \quad \tilde{\psi}\sigma_{\mu\nu}\psi\,, \quad \tilde{\chi}\overline{\sigma}_{\mu\nu}\chi.$$

A suitable candidate for a pair of relativisticly invariant wave equations are

$$\overline{\sigma}^{\mu} p_{\mu}\psi + m\chi + 0$$

$$\sigma^{\mu} p_{\mu}\chi - m\psi = 0$$

Together, these equations imply that

$$\left(p^2 - m^2\right)\psi = 0 = \left(p^2 - m^2\right)\chi$$

For $2+2$ dimensions we employ

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & i\tau^2 \\ -i\tau^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \qquad (A.26)$$

with $g^{\mu\nu} = \mathrm{diag}(+,+,-,-)$. If now

$$C = -A = -i\gamma^3\gamma^4 \qquad (A.27)$$

and $\psi_C = C(\overline{\psi})^T = C(\psi^{\dagger}A)^T = \psi^*$, then it is apparent that a spinor can now be both Majorana ($\psi = \psi_C$) and Weyl ($\psi = \pm\gamma^5\psi$ where $\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$). The 2-component notation we use for $2+2$ dimensions is defined in [5].

8.3 Five Dimensions

In $5+0$ dimensions, we use the matrices of (A.16); the analogues of (A.17) continue to hold.

The following equations are useful

$$\gamma^A\gamma^B\gamma^C = \delta^{AB}\delta^C - \delta^{AC}\gamma^B + \delta^{BC}\gamma^A + \epsilon^{ABCDE}\Sigma^{DE} \tag{A.28}$$

$$\begin{aligned}\gamma^A\gamma^B\gamma^C\gamma^D &= \delta^{AB}\delta^{CD} - \delta^{AC}\delta^{BD} + \delta^{AD}\delta^{BC} \\ &+\epsilon^{ABCDE}\gamma^E - 2\Big[\delta^{AB}\Sigma^{CD} \\ &+\delta^{AC}\Sigma^{DB} + \delta^{BC}\Sigma^{AD} + \delta^{AD}\Sigma^{BC} \\ &+\delta^{BD}\Sigma^{CA} + \delta^{CD}\Sigma^{AB}\Big]\end{aligned} \tag{A.29}$$

$$(\Sigma\cdot A)(\Sigma\cdot B) = -\tfrac{1}{2}A\cdot B + \tfrac{1}{4}\epsilon^{ABCDE}A^{AB}B^{CD}\gamma^E + 2A^{AC}B^{BC}\Sigma^{AB}. \tag{A.30}$$

Just as (A.18) can be used to derive (A.19), we can show

$$\Sigma^{AB}_{ij}\Sigma^{AB}_{k\ell} = -\tfrac{1}{2}\Sigma^{AB}_{i\ell}\Sigma^{AB}_{kj} - \tfrac{1}{4}\gamma^A_{i\ell}\gamma^A_{kj} - \tfrac{5}{4}\delta_{i\ell}\delta_{kj} \tag{A.31}$$

$$\gamma^A_{ij}\gamma^A_{k\ell} = -\tfrac{3}{4}\gamma^A_{i\ell}\gamma^A_{kj} + \tfrac{5}{4}\delta_{i\ell}\delta_{kj} - \tfrac{1}{2}\Sigma^{AB}_{i\ell}\Sigma^{AB}_{kj} \tag{A.32}$$

$$\delta_{ij}\delta_{k\ell} = \tfrac{1}{4}\delta_{i\ell}\delta_{kj} + \tfrac{1}{4}\gamma^A_{i\ell}\gamma^A_{kj} - \tfrac{1}{2}\Sigma^{AB}_{i\ell}\Sigma^{AB}_{kj} \tag{A.33}$$

Together, (A.31) to (A.33) can be used to demonstrate that

$$\Sigma^{AB}_{ij}\Sigma^{AB}_{k\ell} + \Sigma^{AB}_{i\ell}\Sigma^{AB}_{kj} = -\left(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}\right) = -\left(\gamma^A_{ij}\gamma^A_{k\ell} + \gamma^A_{i\ell}\gamma^A_{kj}\right). \tag{A.34}$$

In $5+0$ dimensions we have

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & i\tau^i \\ -i\tau^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.35}$$

and hence it is appropriate to define $\overline{\psi} = \psi^\dagger$ and to have $C = -\gamma^1\gamma^3$ so that

$$\gamma^A = C^{-1}\gamma^{AT}C. \tag{A.36}$$

We hence take $\psi_C = C\overline{\psi}^T$; ψ cannot be Majorana, but may be Dirac or symplectic Majorana. In $4+1$ dimensions we employ

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{A.37}$$

define $\overline{\psi} = \psi^\dagger\gamma^0$ and $\psi_C = C\overline{\psi}^T$ with

$$C = -i\gamma^0\gamma^2\gamma^5 = C^\dagger = C^{-1} = -C^T \tag{A.38}$$

so that $\gamma^A = C^{-1}(\gamma^A)^T C$. Since $(\psi_C)_C = -\psi$, spinors can be either Dirac or symplectic Majorana, but not Majorana.

In $3+2$ dimensions, we have

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.39}$$

so that it is appropriate that we take $\overline{\psi} = \psi^\dagger\left(\gamma^1\gamma^2\gamma^3\right)$ and $\psi_C = C\overline{\psi}^T$ where

$$C = i\gamma^2\gamma^3 \tag{A.40}$$

so that $\gamma^A = C^{-1}(\Gamma^A)^T C$ as in 4+1 dimension. The Majorana condition $\psi = \psi_C$ is consistent as $(\psi_C)_C = \psi$ in $3+2$ dimensions.

An additional identity that is useful in five dimensions is

$$\left(\Sigma^{AB}C\right)_{ij}(\Sigma_{AB})_{k\ell} + \left(\Sigma^{AB}C\right)_{jk}(\Sigma_{AB})_{i\ell} + \left(\Sigma^{AB}C\right)_{ki}(\Sigma_{AB})_{j\ell} = 0; \tag{A.41}$$

this holds in $5+0$ dimensions, $4+1$ dimensions and $(3+2)$ dimensions. It is proven by expanding the left side of (A.41) in the form $P_{ij}\delta_{k\ell} + Q^A_{ij}(\gamma_A)_{k\ell} + R^{AB}_{ij}(\Sigma_{AB})_{k\ell}$; contraction with $\delta_{\ell k}$, $(\gamma_C)_{\ell k}$ and $(\Sigma_{CD})_{\ell k}$ leads to $P = Q^A = R^{AB} = 0$.

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